

## BC Calculus Summer Assignment 2024

### Calculus Topics

Calculus is the study of four topics, organized into three parts: A, B, C

#### **Limits (A)**

Limits are the  $y$ -values toward which graphs or functions converge.

Examples include horizontal asymptotes and the famous irrational numbers  $e$  and  $\pi$ .

#### **Derivatives (A)**

Derivatives are slopes along curves.

Unlike slopes along lines, which stay constant, slopes along curves continually change.

#### **Integrals (B)**

Integrals are areas under curves.

They have broad applications and allow us to calculate the areas and volumes of unconventional shapes.

#### **Series (C)**

Series are sums of terms that follow a pattern.

For example, the numeric series  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$  approaches the value 1.

Series can also include variables, and thus become functions.

### Summer Assignment

The five explorations in this summer assignment introduce the topics described above.

Instead of reviewing what you already know, you will explore and learn new concepts.

#### **Plan Ahead**

Each exploration will likely take about two hours, so plan ahead and start early.

I encourage you to work with classmates, but you must submit your own written work.

Make sure that what you submit reflects your own understanding, and not just someone else's.

#### **Help Sessions**

I will offer several help sessions during the last week of summer break.

Later this summer, check H-B's website for specific times and locations.

#### **Due Dates**

Explorations 1-2 are **due Sunday August 11**.

Save your work as a single pdf file and email it to me: mark.dickson@apsva.us

Explorations 3-5 are **due the first day of school**.

Save your work as a single file pdf and submit it on Canvas the first day of school.

I hope you enjoy these explorations!

Have a terrific summer, and I look forward to a great year together.

Mark

mark.dickson@apsva.us

## Summer Exploration #1: Limits & Derivatives

Do all you work on separate paper, not along the margins of this handout.

In this activity, you will explore how slopes behave along curves.

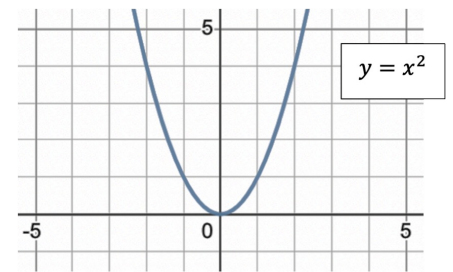
Slopes have important applications. For example:

Slopes along an object's velocity graph represent acceleration.

Slopes along the graph of a virus's total infections represent daily infection rates.

Unlike slopes along lines, which never change, slopes along curves continually change.

Let's begin by exploring the slopes along the function  $y = x^2$  shown to the right.



1. In a few sentences, describe how the slope changes as you move from left to right along this curve. Use these terms in your description: positive, negative, steep, shallow, flat

### Average Slope

The average slope between any two points is this familiar quotient:  $m = \frac{y_2 - y_1}{x_2 - x_1}$   
written in function notation:  $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

For example, along our parabola, the average slope from  $x_1 = 3$  to  $x_2 = 4$  is:  $\frac{4^2 - 3^2}{4 - 3} = \frac{7}{1} = 7$

2. In similar fashion, calculate the average slope from  $x_1 = 3$  to  $x_2 = 3.5$   
Repeat two more times: from  $x_1 = 3$  to  $x_2 = 3.1$  and from  $x_1 = 3$  to  $x_2 = 3.01$

### Instantaneous Slope is a Limit

If we could move  $x_2$  infinitely close to  $x_1$ , we could calculate the "instantaneous slope" at  $x = 3$ .

Unfortunately, that calculation leads to this problematic quotient:  $\frac{3^2 - 3^2}{3 - 3} = \frac{0}{0}$

So, we must take a different approach.

3. The instantaneous slope at  $x = 3$  is a limiting value as we move  $x_2$  closer and closer to  $x_1 = 3$ .

Let  $h$  represent that tiny difference between the two  $x$ -values:  $x_2 - x_1 = h$

So, mathematically, we want to calculate this limit:  $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h}$

### Graphical Solution

4. On your calculator, graph the quotient  $\frac{(3+h)^2 - 3^2}{h}$ . Of course, you will have to type  $x$  instead of  $h$ .

This quotient's key characteristic is a hole at  $h = 0$ .

To see that hole on your calculator's graph, you will have to turn off the  $y$ -axis. Do that in FORMAT, Axes Off

Draw an accurate sketch of the graph. Include the hole as an open circle.

At what  $y$ -value is the hole located?

5. Look at the table on your calculator:  
In "table set", start the table at  $h = -0.5$ , and increment by  $\Delta h = 0.1$ . (Remember, your calculator's  $x$  represents  $h$ .)  
In the table, you'll see an error at  $h = 0$ .  
What value does the quotient seem to approach as you move closer and closer to  $h = 0$ ?  
Is that consistent with your graph? Explain.

## Algebraic Solution

6.

Let's now use algebra to evaluate:  $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h}$

First, expand (remember to FOIL) and simply the numerator. Your resulting quotient should now be:  $\lim_{h \rightarrow 0} \frac{6h+h^2}{h}$

Next, factor the numerator, and then simplify the quotient. The act of simplifying removes the hole at  $h = 0$ .

Finally, re-substitute  $h = 0$ .

Is your final result consistent with your earlier graphical solution? Great!

7.

Let's now use algebra to determine the slope at several other locations along  $y = x^2$ .

For example, here is the slope at  $x = -7$ :

$$\lim_{h \rightarrow 0} \frac{(-7+h)^2 - (-7)^2}{h} = \lim_{h \rightarrow 0} \frac{49 - 14h + h^2 - 49}{h} = \lim_{h \rightarrow 0} \frac{-14h + h^2}{h} = \lim_{h \rightarrow 0} \frac{h(-14+h)}{h} = \lim_{h \rightarrow 0} \frac{-14+h}{1} = \frac{-14}{1} = -14$$

In similar fashion, calculate the slopes at these other locations:  $x = -4$ ,  $x = -2$ ,  $x = 0$ ,  $x = 5$ ,  $x = 8$

8.

Perhaps you have discovered a pattern to how the slope relates to the  $x$ -value.

Without any calculating, what are the slopes at  $x = -18$  and at  $x = 19$ ?

Generalize your results: what is the slope at any  $x$ ?

9.

Now, prove that generalized result: use algebra to evaluate:  $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$

Does that confirm your earlier work? Terrific!

## Derivative

The instantaneous slope along a curve is called the "derivative".

It's written mathematically two different ways:  $f'(x)$  or  $\frac{dy}{dx}$

The first expression is pronounced "f prime of  $x$ ".

So, your work so far has shown that: if  $f(x) = x^2$ , then  $f'(x) = 2x$

Written another way: if  $y = x^2$ , then  $\frac{dy}{dx} = 2x$

## Cubic Functions

Let's now apply limits and algebra to explore slopes along this cubic function:  $f(x) = x^3$

Doing so will require expanding cubic binomials properly:  $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

10.

Here is the slope at  $x = -4$ :

$$\lim_{h \rightarrow 0} \frac{(-4+h)^3 - (-4)^3}{h} = \lim_{h \rightarrow 0} \frac{(-64 + 48h - 12h^2 + h^3) + 64}{h} = \lim_{h \rightarrow 0} \frac{48h - 12h^2 + h^3}{h} = \lim_{h \rightarrow 0} \frac{h(48 - 12h + h^2)}{h} = \lim_{h \rightarrow 0} \frac{48 - 12h + h^2}{1} = \frac{48}{1} = 48$$

In similar fashion, calculate the slope at these other locations:  $x = -2$ ,  $x = 0$ ,  $x = 1$ ,  $x = 5$

11.

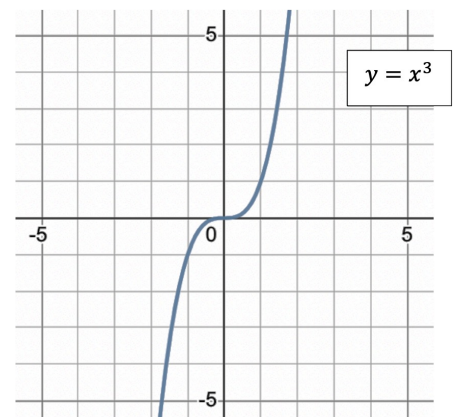
Perhaps you notice the following about your slopes: each is three times a perfect square.

So, without calculating at all, what are the slopes at  $x = -5$  and at  $x = 6$ ?

Generalize your results: what is the slope at any  $x$ ?

12.

Now prove that result: use algebra to evaluate:  $\lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$



## Power Rule

Let's summarize what you've learned so far: if  $f(x) = x^2$  then  $f'(x) = 2x$

if  $f(x) = x^3$  then  $f'(x) = 3x^2$

We can generalize this with the "power rule": if  $f(x) = x^n$  then  $f'(x) = nx^{n-1}$

That exponent can be any type of number: positive, negative, even a fraction.

Example with a negative exponent: if  $f(x) = \frac{1}{x^4}$  then  $f'(x) = \frac{-4}{x^5}$

Here's why:

Rewrite the function:  $f(x) = \frac{1}{x^4} = x^{-4}$

Apply the power rule:  $f'(x) = -4x^{-5}$

Rewrite the derivative:  $f'(x) = \frac{-4}{x^5}$

Example with a fractional exponent: if  $f(x) = \sqrt[3]{x}$  then  $f'(x) = \frac{1}{3x^{\frac{2}{3}}}$

Here's why:

Rewrite the function:  $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

Apply the power rule:  $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

Rewrite the derivative:  $f'(x) = \frac{1}{3x^{\frac{2}{3}}}$

13.

Now you try:

Use the power rule to write the derivatives of the following functions:

$$f(x) = \frac{1}{x^7} \quad g(x) = \sqrt[4]{x} \quad w(t) = \frac{1}{\sqrt{t}}$$

14.

Do some quick research on Archimedes, Isaac Newton, and Gottfried Leibniz.

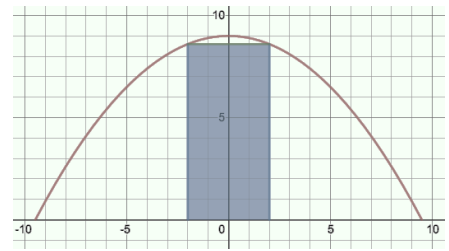
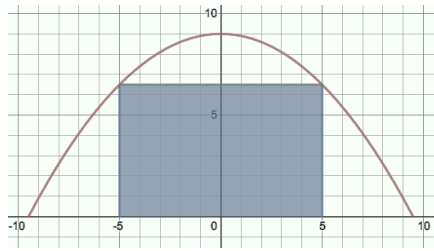
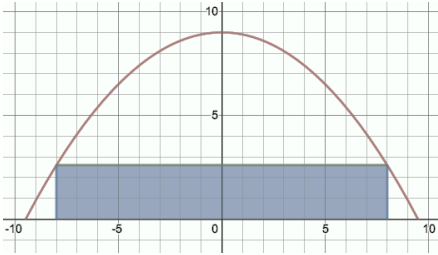
For each, explain in a sentence or two how this exploration relates to his mathematical work.

## Summer Exploration #2: Maximize a Function

Present your work on separate paper, not along the margins of this handout.

In this exploration, you will calculate the areas of rectangles inscribed in the parabola  $f(x) = 9 - \frac{1}{10}x^2$

Shown below are three such rectangles:



Notice that each rectangle is symmetric about the  $y$ -axis.

As the first diagram shows, a large  $x$ -value (in the lower-right corner) produces a short, wide rectangle.

As the third diagram shows, a small  $x$ -value (in the lower-right corner) produces a tall, thin rectangle.

### Steps

1.  
For each rectangle shown, determine its lower-right-hand-corner  $x$ -value, its length, its height, and its area. Don't estimate height. Instead use the parabolic function to find each height precisely. For example, in the first diagram, the rectangle's height is:  $9 - \frac{1}{10}8^2 = 2.6$   
Record your results in a four-column chart.

2.  
Repeat step 1 for several other  $x$ -values of your choosing and include them as more rows in your chart. Choose  $x$ -values that you think might result in large rectangular areas. Try to converge on the  $x$ -value that might maximize that area.

3.  
Sketch on graph paper a plot of rectangular areas (from column 4) as a function of  $x$ -values (from column 1). The horizontal axis should represent  $x$ -values; the vertical axis should represent areas. Label your axes & place values on them.

4.  
Generalize your calculations for *any*  $x$ -value. In other words, express the rectangular area as a function of  $x$ . Graph that area function on your calculator. Use a window similar to that from your hand-drawn graph.

5.  
Trace along that area function and also look at its table. Start your table at  $x = 0$  and increment by small amounts, such as  $\Delta x = 0.1$ . Do your earlier calculations appear? They should!

6.  
Find the  $x$ -value that maximizes the area function. Do this by using Calc-Maximum while viewing the graph. Record the  $x$ -value and the resulting maximum area.

7.  
Here's a more elegant way to find that  $x$ -value:

At a maximum or minimum, a function's slope (derivative) equals zero.

So, use the power rule to differentiate your area function. You'll first need to re-write that function by distributing the  $2x$ .

Next, set the derivative equal to zero, and then solve for  $x$ .

Does your result confirm your earlier work? Terrific!

### Summer Exploration 3: Driving Project

Present your work on separate paper, not along the margins of this handout.

This project involves taking a 30-minute car ride, collecting data, and analyzing results. You may collect data with another student, but each of you must submit your own write-up.

#### Collect Data

- Record your car's odometer reading **before starting your car ride**.
- With another person driving, take a 30-minute car ride in a residential area. Find a route that allows you to change speeds throughout the trip.
- During the trip, record your velocity every two minutes (every  $\frac{1}{30}$ th of an hour).  
When done, you should have sixteen velocities recorded at these times:  $t = 0, \frac{1}{30}, \frac{2}{30}, \frac{3}{30}, \dots, \frac{15}{30}$  hours.
- Record the odometer reading **after completing your car ride**.

#### Graph and Calculations

1. Present your data in a table, and plot velocity vs. time (in hours) on graph paper. Be sure that velocity is on the vertical axis and time is on the horizontal axis. Include units and increments on each axis. } Use hours, not minutes!

2. Connect your sixteen data points with line segments to form your velocity graph.

3a. Working from left to right, calculate the area under your velocity graph for each  $\frac{1}{30}$  time interval. You will calculate fifteen areas. Some might be rectangles or triangles, and many will likely be trapezoids. In case you don't remember the area of a trapezoid:

$$A_{trap} = \frac{1}{2}(b_1 + b_2)h \text{ where the bases are the parallel sides.}$$

3b. Add your fifteen areas to determine the total area under your velocity graph.

#### Analysis

4. In a well-written paragraph, answer these questions:

- a) How does the total area under your velocity graph compare to the two odometer readings?
- b) What does the area under your velocity graph represent? Address units.
- c) Divide the total area by  $\frac{1}{2}$  hour, which is the total time you spent driving. What does that result represent? Again, address units.
- d) What does the slope during each time interval represent? Address units. What does a negative slope imply about how you were moving during that time interval?
- e) Is your velocity data a complete picture of your velocity during your trip? If yes, explain why. If no, explain how you could make it more complete.

## Integrals

Areas under curves are known as “integrals”.

Integrals have important applications. For example:

The integral of an object’s velocity curve represents how far it has traveled.

The integral of a virus’s infection rate represents the total number of people infected.

### Work with a Function

Let’s now work with the function  $f(x) = x^2$  shown to the right.

We will approximate the shaded region’s area along the interval  $2 \leq x \leq 8$

5.

First, approximate that area by counting squares in the shaded region.

Don’t get hung up on precision. A rough estimate is fine.

Notice that the area of one square does not equal one. Account for that.

6.

Next, approximate that same area, but now by using the four methods shown on the next page.

Don’t approximate any  $y$ -values! Use the function to get precise  $y$ -values.

7.

For each of the four methods, explain whether your result under- or over-estimates the true area under the curve.

8.

Now consider, just for a moment, the function shown to the right.

Repeat the previous step.

Don’t do any calculating! For each of the four methods, just explain.

9.

Let’s now return to  $f(x) = x^2$  and the shaded region along the interval  $2 \leq x \leq 8$

Repeat your left-hand calculation, but now use six rectangles rather than just three.

Try to do this without actually drawing the rectangles.

10.

How does this new result with six rectangles compare to your earlier result with just three rectangles?

Without doing any calculations, explain how using 100 rectangles would compare to using six or three rectangles.

To arrive at the true value of the shaded region’s area, how many rectangles would we have to use?

11.

In this activity, you have calculated individual rectangular areas by multiplying  $y$ -values by changes in  $x$ :  $f(x) \cdot \Delta x$

Then, you summed those individual rectangles to approximate the value of the shaded region’s area:  $\sum f(x) \cdot \Delta x$

The true value of that area is a limit: the result of using infinitely-many, infinitely-thin rectangles:  $\int_2^8 f(x) \cdot dx$

12.

This year, we will learn elegant ways to calculate integrals (areas under curves) without relying on trapezoids and rectangles.

Perhaps you’re eager to know the exact value of the shaded region’s area in this exploration:

Place the function in your calculator:  $y_1 = x^2$

View the graph in a good window:  $0 \leq x \leq 10, 0 \leq y \leq 100$

Select CALC from the top of the keypad.

Select the symbol  $\int f(x) dx$

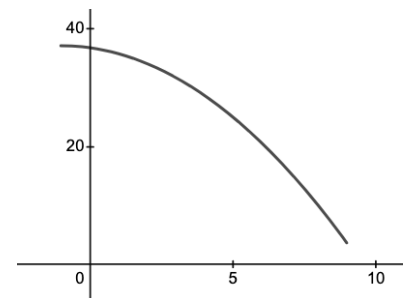
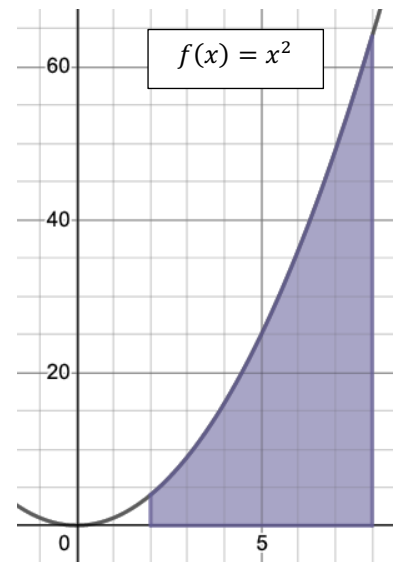
For the lower limit, type “2”, and for the upper limit, type “8”

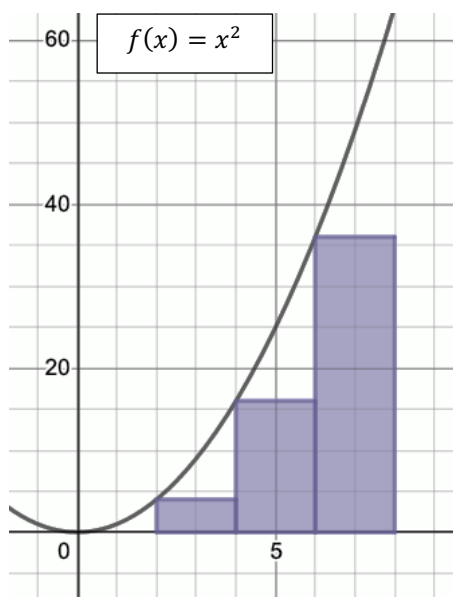
Your calculator will then shade the region and provide the exact value.

13.

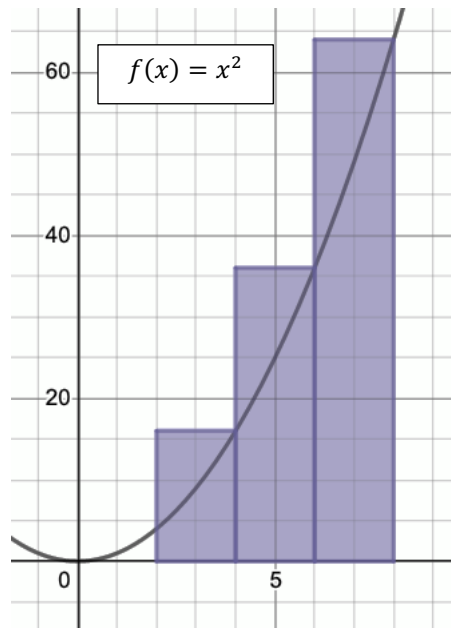
Rectangles that approximate areas under curves are known as “Riemann Sums”.

Do some quick research on Bernhard Riemann and describe four interesting facts about him.

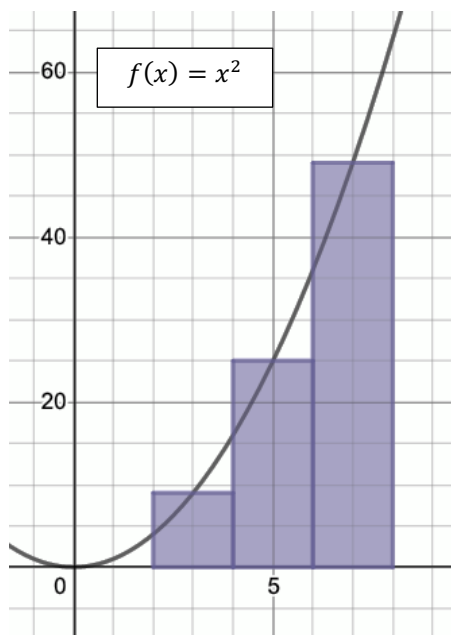




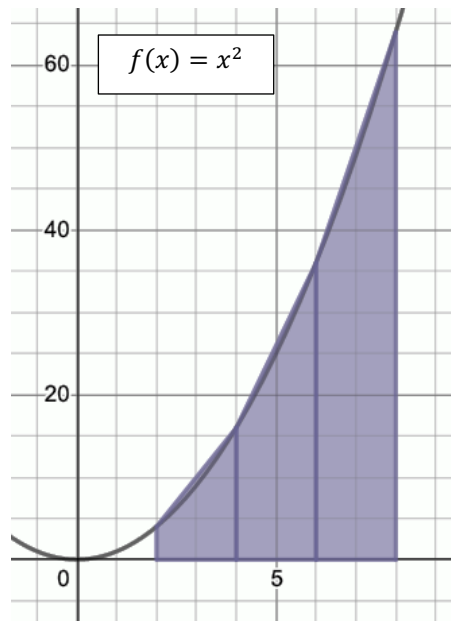
**Three Left-Hand Rectangles**  
 Each rectangle's height is determined by the  $y$ -value at the left edge of its  $x$ -interval.



**Three Right-Hand Rectangles**  
 Each rectangle's height is determined by the  $y$ -value at the right edge of its  $x$ -interval.



**Three Midpoint Rectangles**  
 Each rectangle's height is determined by the  $y$ -value at the midpoint of its  $x$ -interval.



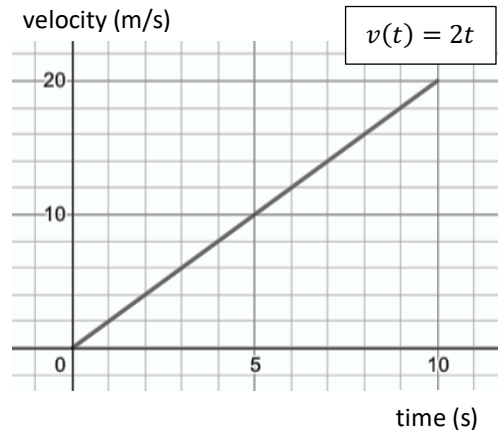
**Three Trapezoids**



## Summer Exploration #4: Integrals More Elegantly

Do all your work on separate paper, not along the margins of this handout.

Consider an object that accelerates from zero to twenty meters per second. It takes ten seconds to do so, as shown by the graph.



### Calculate Area with Geometry

1. Explain why each square on this graph paper represents two meters. Address both the value "2" and unit "meters".

The object's displacement is the area under this velocity function. Because this function is linear, not curved, we can count squares or use geometry to calculate exact areas. For example, the displacement from time  $t = 0$  to time  $t = 4$  equals 16 meters.

Written mathematically:  $\int_0^4 2t \, dt = 16$

Verbally: "The integral of  $2t$  from zero to four equals sixteen."

2. Copy onto your own paper the table shown. Complete the entries, using the graph to calculate displacements.
3. Perhaps you see a pattern to your results. In a sentence, explain how the displacement relates to the time interval's endpoint.

time interval (seconds)	notation	displacement (meters)
$0 \leq t \leq 0$		
$0 \leq t \leq 1$		
$0 \leq t \leq 2$		
$0 \leq t \leq 3$		
$0 \leq t \leq 4$	$\int_0^4 2t \, dt$	= 16
$0 \leq t \leq 5$		
$0 \leq t \leq 6$		
$0 \leq t \leq 7$		
$0 \leq t \leq 8$		
$0 \leq t \leq 9$		
$0 \leq t \leq 10$		

4. Let's now use our results in the table to calculate other displacements:

For example:  $\int_5^8 2t \, dt = 64 - 25 = 39$  meters

So, in the time interval  $5 \leq t \leq 8$ , the object moved 39 meters.

Calculate these displacements. Be sure to show the subtraction step.

$$\int_3^6 2t \, dt =$$

$$\int_2^9 2t \, dt =$$

$$\int_7^8 2t \, dt =$$

6. Generalize your results for any time interval  $a \leq t \leq b$ :

$$\int_a^b 2t \, dt =$$

### Anti-Derivatives

As you've just shown, the integral of  $2t$  involves evaluating  $t^2$ . That's a powerful discovery, because  $2t$  is the derivative of  $t^2$ . Said another way:  $t^2$  is the "anti-derivative" of  $2t$ .

7. Let's now apply this discovery to calculate integrals under a new function:  $f(x) = x^2$ . This function is curved, not linear, so counting squares will no longer provide exact areas. So instead, let's use the anti-derivative. In a sentence, explain why the anti-derivative of  $x^2$  is  $\frac{1}{3}x^3$ . Generalize: what is the anti-derivative of  $x^n$ ?

8.

Let's now calculate some integrals under the function  $f(x) = x^2$

Here are two examples:

$$\int_2^5 x^2 dx = \left[ \frac{1}{3}x^3 \right]_2^5 = \left( \frac{1}{3}5^3 \right) - \left( \frac{1}{3}2^3 \right) = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = 39$$

$$\int_1^6 x^2 dx = \left[ \frac{1}{3}x^3 \right]_1^6 = \left( \frac{1}{3}6^3 \right) - \left( \frac{1}{3}1^3 \right) = \frac{216}{3} - \frac{1}{3} = \frac{215}{3} = 71.\bar{6}$$

Now you try. Show all the steps, just like in the above examples.

$$\int_0^7 x^2 dx =$$

$$\int_4^9 x^2 dx =$$

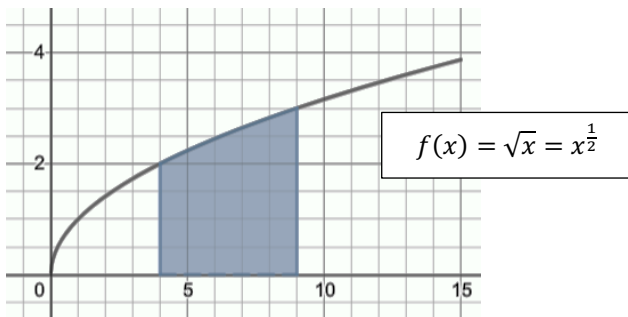
$$\int_2^8 x^2 dx =$$

Do you recognize this last result? It's the area from your previous exploration!

### Other Functions

Let's conclude by applying what you've learned to some new functions.

Here's an example that involves a fractional exponent:

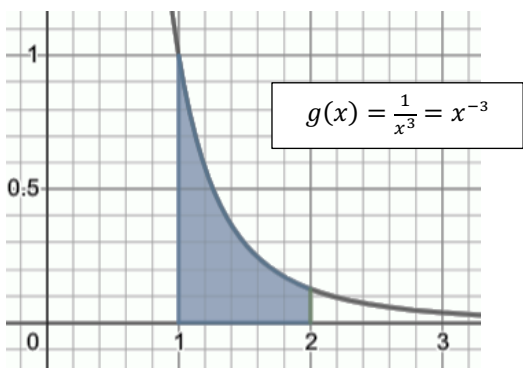


anti-derivative:  $\frac{2}{3}x^{\frac{3}{2}}$

integral:  $\int_4^9 \sqrt{x} dx = \left[ \frac{2}{3}x^{\frac{3}{2}} \right]_4^9 = \left( \frac{2}{3}9^{\frac{3}{2}} \right) - \left( \frac{2}{3}4^{\frac{3}{2}} \right) = \frac{2}{3}27 - \frac{2}{3}8 = \frac{54}{3} - \frac{16}{3} = \frac{38}{3} = 12.\bar{6}$

check by counting squares: about 24 squares, each with area =  $\frac{1}{2}$ , so about 12. ✓

Here's an example that involves a negative exponent:



anti-derivative:  $\frac{-1}{2}x^{-2} = \frac{-1}{2x^2}$

integral:  $\int_1^2 \frac{1}{x^3} dx = \left[ \frac{-1}{2x^2} \right]_1^2 = \left( \frac{-1}{2 \cdot 2^2} \right) - \left( \frac{-1}{2 \cdot 1^2} \right) = \frac{-1}{8} - \left( \frac{-1}{2} \right) = \frac{-1}{8} + \frac{1}{2} = \frac{3}{8} = .375$

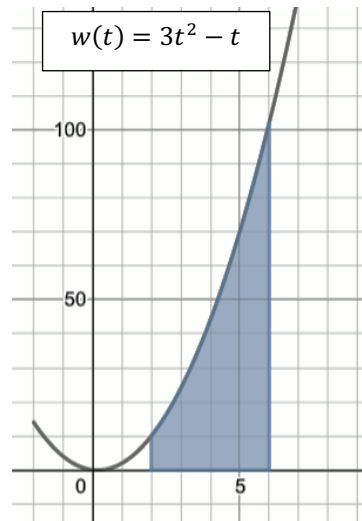
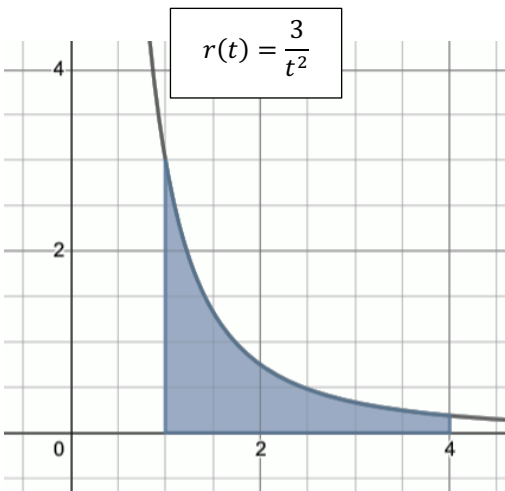
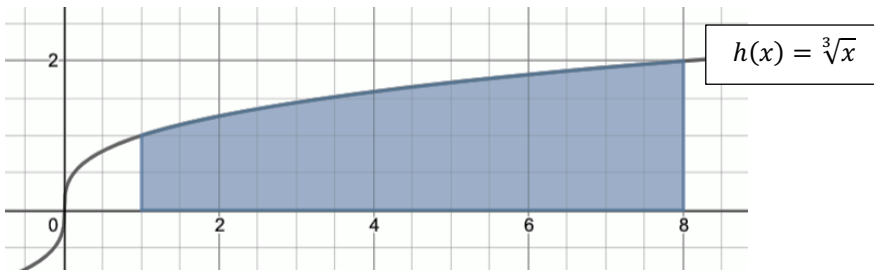
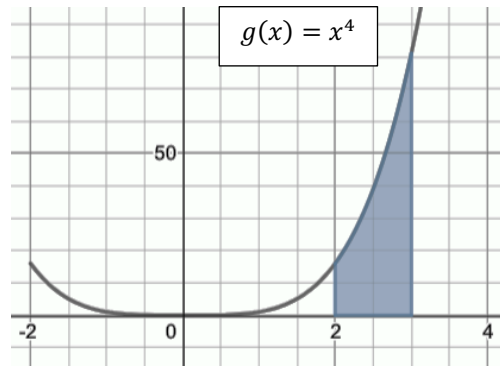
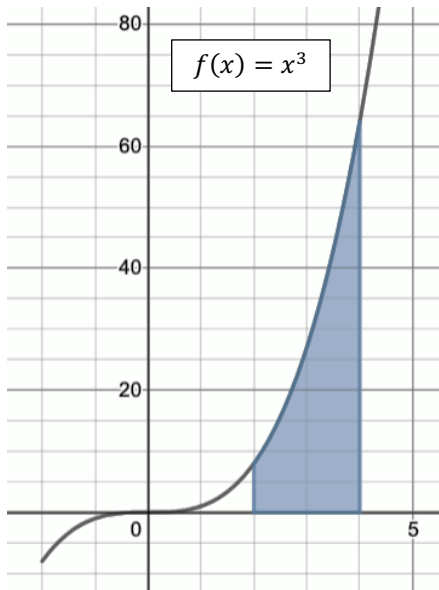
check by counting squares: about 18 squares, each with area = .02, so about 0.36. ✓

9.

Calculate the areas for the graphs on the next page.

In each case, provide the anti-derivative, the integral calculation with all steps shown, and the check by counting squares.

Good luck!



## Summer Exploration #5: Introduction to Series

Present your work on separate paper, not along the margins of this sheet.

In Algebra 2, you learned about sequences & series.

In a geometric series, the first term is  $a$ , and the common ratio is  $r$ .

If that common ratio lies between -1 and 1, then the following is true:

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + ar^4 + \dots \quad \text{interval: } -1 < r < 1$$

### Work with Numbers

1.

Replace  $a$  with 8 and  $r$  with  $\frac{1}{2}$ . Evaluate the quotient  $\frac{a}{1-r}$ .

Show that, as you use more & more terms, the series  $a + ar + ar^2 + ar^3 + ar^4 + \dots$  approaches that same value. Wow! Will the series ever actually reach that value? Explain your reasoning.

2.

Replace  $a$  and  $r$  with other values of your choosing. Be sure that  $r$  lies in the interval:  $-1 < r < 1$

Does the series still approach the quotient? Terrific!

### Work with Functions & Graphs

3.

This time, replace  $a$  with 1 and  $r$  with  $x$ .

Write down the resulting equation and its interval.

Unlike a numeric series, this is a function series; it contains the variable  $x$  rather than just numbers.

In this case, the series models the function  $\frac{1}{1-x}$ .

4.

Graph on your calculator each side of your equation. Use  $y_1$  for the quotient and  $y_2$  for the series.

Since you can't type the entire series, go out to just the sixth-degree term ( $x^6$ ).

A good viewing window is  $-5 \leq x \leq 5$  and  $-2 \leq y \leq 2$ .

5.

Sketch the result and identify which curve is the function and which is the series.

Look at the two graphs.

Do they appear to converge in the interval  $-1 < x < 1$ ? They should!

6.

Look at your calculator's table of values. Start at  $x = 0$  with  $\Delta x = 0.1$

Compare the values from the series ( $y_1$ ) to those on the function ( $y_2$ ).

Describe how the accuracy of the series changes as you move away from  $x = 0$ .

Be sure to look in both the positive & negative directions.

Describe what happens to the accuracy when you move outside the interval  $-1 < x < 1$

7.

Add the next three terms to your series in  $y_2$ , so that it's now a 9<sup>th</sup> degree expression.

Look again at the table and describe how the accuracy of the series has changed.

Explain why that makes sense.

8.

Now replace  $a$  with 1 and  $r$  with  $x - 3$  in the above equation and in the interval.

You now have the series for  $\frac{1}{4-x}$  and the interval on which it converges.

9.

Graph both the function and its series, this time in  $y_3$  and  $y_4$ . Go out to the 6<sup>th</sup> degree.

Turn off  $y_1$  and  $y_2$ . Also, extend your  $x$ -window to see  $y_3$  and  $y_4$  better.

Does the resulting graph confirm your work in the previous step? It should!

Sketch the graph and identify which curve is the function and which is the series.

### Trigonometric Series

10.

One famous infinite series in mathematics is:  $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

Notice that it has only odd powers and that it alternates signs.

It also includes the factorial symbol, which you can find on your calculator under "math" "probability".

Graph this series in  $y_5$ . Go out to the thirteenth degree ( $\frac{1}{13!}x^{13}$ ).

Turn off your other functions and use the window:  $-3 \leq x \leq 3, -2 \leq y \leq 2$

Sketch the resulting graph.

11.

Which trigonometric function does this series appear to model near the origin?

Test your conjecture in  $y_6$ . Were you correct? Terrific!

12.

Look at your calculator's table of values for  $y_5$  and  $y_6$ . Start at  $x = 0$  with  $\Delta x = 0.1$

How does the accuracy of this series change as you move away from  $x = 0$ ?

13.

Add the next three terms to your series, so that it now goes out to the 19<sup>th</sup> degree.

Look again at the table of values. How did the accuracy change?

Explain why that makes sense.

### Transforming Series

You have just seen that:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

The more terms you use, the more accurate the result.

Let's now transform this series, just as you have done in past math courses with traditional functions.

Here are three examples:

horizontal compression:  $\sin(2x) = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7 + \dots$

vertical shift:  $4 + \sin x = 4 + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

horizontal shift:  $\sin(x - 1) = (x - 1) - \frac{1}{3!}(x - 1)^3 + \frac{1}{5!}(x - 1)^5 - \frac{1}{7!}(x - 1)^7 + \dots$

14.

Now you try: create the series for  $g(x) = -1 + \sin\left(\frac{1}{2}x\right)$ . Go out to the 13<sup>th</sup> degree.

In a sentence, explain how the values  $\frac{1}{2}$  and  $-1$  transform both the function and its series.

15.

Graph both the function and the series.

Use  $y_7$  and  $y_8$ . Your current window is fine:  $-3 \leq x \leq 3, -2 \leq y \leq 2$

Are the graph and the table consistent with your expectations? Explain.

## Derivatives

16.

Use the power rule to differentiate, term-by-term, the right-hand side of the equation:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

For example, the derivative of  $\frac{1}{3!}x^3$  is  $\frac{1}{3!}3x^2$ , which simplifies to  $\frac{1}{2!}x^2$

17.

Place your result in  $y_9$ . Go out to the 12<sup>th</sup> degree.

What new trig function does it model? Wow!

Place that new trig function in  $y_{10}$

Sketch the resulting graphs of  $y_9$  and  $y_{10}$  and explain what you see, both on the graph and in the table.

18.

Another famous series in mathematics is:  $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$

Place this series in  $y_{11}$ . Go out to the seventh degree.

Discover what function it models by doing the following:

Sketch its graph in the window:  $-2 \leq x \leq 3$  and  $-5 \leq y \leq 20$

Use the table to evaluate the series at  $x = 0$  and  $x = 1$ .

What function do you think this series is modeling? Wow!

Differentiate the series term-by-term. What results? Wow, again!

## Summary

You have just built series for four functions:

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + ar^4 + \dots \quad \text{This works only in the interval } -1 < r < 1$$

$$\left. \begin{aligned} \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \\ e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \end{aligned} \right\} \text{These work for all } x\text{-values.}$$

These series are perfectly accurate at  $x = 0$ , but they lose accuracy as we move away from  $x = 0$ .

To improve accuracy, simply use more terms!

We can transform these series to build other series. For example:

$$\begin{aligned} xe^{3x^2} &= x \cdot [1 + (3x^2) + \frac{1}{2!}(3x^2)^2 + \frac{1}{3!}(3x^2)^3 + \frac{1}{4!}(3x^2)^4 + \dots] \\ &= x + 3x^3 + \frac{1}{2!}9x^5 + \frac{1}{3!}27x^7 + \frac{1}{4!}81x^9 + \dots \end{aligned}$$

19.

Now you try: build the series for  $\frac{e^{x^3}-1}{x^2}$  this way:

First, take the  $e^x$  series and replace every  $x$  with  $x^3$ .

Next, subtract 1 from that series. (This just removes the first term.)

Finally, divide every term by  $x^2$ .

Did you arrive at this result?  $\frac{e^{x^3}-1}{x^2} = x + \frac{1}{2!}x^4 + \frac{1}{3!}x^7 + \frac{1}{4!}x^{10} + \dots ?$

Terrific!

20.

Do some quick research on Brook Taylor and Colin Maclaurin.

In a few sentences, explain how this exploration relates to their mathematical work.